Simulation of the Quasi-Steady Behavior of Under-Expanded Supersonic Jets due to Instability Waves

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ABSTRACT

A time accurate numerical study was carried out to study the quasi-steady behavior of under-expanded jets. The quasi-steadiness is caused by interaction of the large scale instabilities with the shock cell structure of the under-expanded jet. A time marching TVD scheme along with a corrected $k - \varepsilon$ turbulence model, which accounts for compressibility and turbulence/shock wave interaction, was used for simulations. Then, linear instability theory was applied to the same jet to explain and support the numerical calculations. It was shown that the time dependent fluctuations in flow parameters were maximum in the vicinity of shock waves. In addition, two types of instability waves, corresponding to two distinct frequency ranges, were found to exist and interact with the flow structure.

1 BACKGROUND AND INTRODUCTION

The fouling of heat transfer surfaces in Kraft recovery boilers is a significant concern for the pulp and paper industry. The usual approach to controlling fouling is the use of so-called “sootblowers,” that utilize boiler steam to generate supersonic steam jets that are literally used to knock deposits off of the boiler tubes. Sootblower nozzles are nominally designed such that the steam is expanded to ambient pressure at the nozzle exit. But in reality, sootblower jets never perform at exactly the design condition, and so the jet pressure at the nozzle exit is never exactly the ambient pressure inside the boiler.

When the nozzle exit pressure of a supersonic jet is higher than the ambient pressure, shock and expansion waves form, the so-called multi-cell shock structure, through which the pressure of the flow field drops to the ambient value. These jets are usually referred to as “under-expanded”. Figure 1 shows a Schlieren flow visualization of an under-expanded jet by Panda [1]; the nozzle exit is on the left hand side.

The interaction of large scale flow instabilities with shock and expansion waves is one of the important physical phenomena of under-expanded jets. This interaction leads to time-dependent oscillating (quasi-steady) behavior of the flow field. This physical phenomenon is known and has been observed experimentally (e.g. Refs. [1, 2]). Although simulation of under-expanded jets is challenging, given the complicated interaction of shock waves and turbulent mixing, quasi-steady behavior of these flows seems to be captured well by the present simulations.

Here, time accurate simulations are used to study the quasi-unsteadiness of the under-expanded jet of Seiner and Norum [3], with an exit to ambient pressure ratio of 1.45 and an exit Mach number of 2.0, both in the time and frequency domains. The measurements of Seiner and Norum are frequently cited and used to assess numerical models (other measurements by Norum and Seiner [15], for a similar under-expanded jet, confirm the validity of this data). Then, linear jet in-
stability theory [4] is applied to the same jet, and the results are used to explain and support the numerical study. All of the simulations were 2-D axisymmetric, and so the azimuthal instability modes were not captured; given the fine meshes required to capture the shock wave phenomena, 3-D simulations were not attempted.

2 Existence of Instability Waves

The linear instability theory of Tam and Hu [4] for a round compressible jet is used to explain some of the numerical results of the present study. This theory is based on a solution of the linearized Euler equations. The linear solution to an axisymmetric jet yields relations for pressure perturbations inside and outside the jet, respectively, as

\[ p_i = [H_0^{(1)}(i\eta_i, D/2)]J_0(\eta_i r)/J_0(\eta_i D/2)] \exp[i(\alpha x - \omega t)] \]  

(1)

and

\[ p_o = H_0^{(1)}(i\eta_o r) \exp[i(\alpha x - \omega t)] \]  

(2)

where \( H_0^{(1)} \) is the zeroth-order Hankel function of the first kind, \( J_0 \) is the zeroth-order Bessel function of the first kind, \( D \) is the nozzle exit diameter, \( \alpha \) is the wave number, \( \omega \) is frequency, and \( x, r \) and \( t \) are the axial and radial coordinates and time, respectively. Also, \( \eta_o = (\alpha^2 - \omega^2/a_o^2)^{1/2} \) and \( \eta_i = [(\omega - U_o \alpha)^2/a_o^2 - \alpha^2]^{1/2} \), where \( a_o \) is the ambient speed of sound, and \( a_r \) and \( U_r \) are the speed of sound and the flow velocity at the nozzle exit, respectively. Setting the inside and outside solutions equal to each other at the jet boundary, one obtains

\[ DS(\omega, \alpha) \equiv \frac{\eta_o}{\rho_o \omega^2} J_0(\eta_i D/2) H_0^{(1)}(i\eta_o D/2) - \frac{\eta_i}{\rho_o(\omega - U_o \alpha)^2} H_0^{(1)}(i\eta_o D/2) J'_0(\eta_i D/2) = 0 \]  

(3)

where \( DS(\omega, \alpha) \) is usually referred to as the dispersion function. Solutions of \( DS \) yield the possible frequencies, \( \omega \), for a given wave number, \( \alpha \). These are referred to as instability modes. 

This instability theory was applied to the jet studied here, and the solutions of the dispersion function were plotted in the \( \omega, \alpha \) plane, as can be seen in Figure 2, where \( \omega_r \) and \( \omega_i \) represent the real and imaginary parts of \( \omega \) respectively, and \( \alpha D = 30 \). The blue and red lines represent the zeros of the real and imaginary parts of \( DS \), respectively; the intersections are the solutions of \( DS \), i.e. the instability modes. By looking at the pressure eigenfunctions, Tam and Hu [4] argued that these instability modes correspond to two different categories: the Kelvin-Helmholtz and acoustic instability modes, respectively. Carrying out the calculations for different values of \( \alpha \), including complex values, yielded similar results. These confirm the existence of instability modes in the jet flow and prove useful, when the simulation results are studied in the frequency domain.

3 Numerical Method

CFDLib 3.02, a Computational Fluid Dynamics code developed at the Los Alamos National Laboratory, was used for calculations. To calculate the motion of a compressible flow, the Favre averaged equations of conservation of mass, momentum and energy must be solved,

\[ \dot{\rho} \tilde{v} = \frac{\partial \tilde{u}_i}{\partial x_j} \]  

(4)

\[ \dot{\rho} \tilde{u}_i = - \frac{\partial \tilde{p}}{\partial x_j} + \frac{\partial \tilde{v}_j}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_j} \]  

(5)

\[ \dot{\rho} \tilde{E} = \frac{\partial \tilde{u}_j \tilde{u}_j}{\partial x_j} + \frac{\partial \tilde{u}_j \tilde{v}_j}{\partial x_j} + \tilde{u}_j \frac{\partial \tilde{u}_j}{\partial x_j} \]
The turbulence model is the standard $k - \varepsilon$ model, with modifications applied to account for the effects of structural compressibility [5], realizability [6], and shock unsteadiness [7]. The turbulence kinetic energy $k$ and dissipation $\varepsilon$ equations are [8]

$$\frac{\partial \bar{u}_i}{\partial x_j} + q_{Tj}^i + \bar{\rho} \varepsilon = \lambda_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \bar{\rho} \varepsilon + \frac{\partial}{\partial x_j} \left( \mu + \frac{\mu_T}{\sigma} \right) \frac{\partial k}{\partial x_j}$$  

(7)

along with an equation of state $\bar{p} = \bar{\rho} R \bar{T}$, where $\bar{p}$ is density, $\bar{u}_i$ is the velocity vector, $x_j$ is the coordinate vector, $p$ is pressure, $t_{ij}$ is the molecular stress tensor, $\lambda_{ij}$ is the Reynolds stress tensor, $E$ is specific total energy, $q_{Lj}$ and $q_{Tj}$ are the laminar and turbulent heat flux vectors, respectively. $\varepsilon$ is the turbulence dissipation rate, $T$ is temperature, and $\cdots, \cdots, \cdots$ represent the Reynolds average, Favre average and Lagrangian derivative, respectively.

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$$\bar{\rho} \varepsilon = C_{\varepsilon \mu} \beta_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - C_{\varepsilon 2} \bar{\rho} \varepsilon^2 + \frac{\partial}{\partial x_j} \left[ \left( \mu + \frac{\mu_T}{\sigma} \right) \frac{\partial \varepsilon}{\partial x_j} \right]$$  

(8)

where $C_{\varepsilon 1} (=1.44)$, $C_{\varepsilon 2} (=1.92)$, $\sigma_k (=1.0)$ and $\sigma_\varepsilon (=1.3)$ are constant closure coefficients, and from the dynamic eddy viscosity $\mu_T = C_\mu \bar{\rho} \varepsilon^2$, $C_\mu$ is an additional coefficient.

To account for the structural compressibility, Heinz [5] suggested that

$$C_\mu = 0.07 \exp(-0.4 M_s g)$$  

(9)

where $M_s = \frac{S_k}{U_1}$ is the gradient Mach number, and characterizes the strength of compressibility [9]; $S_k = \frac{\partial \bar{u}}{\partial x} \cdot \frac{\partial \bar{u}}{\partial x}$ is the mean shear rate, $U_1$ and $x_2$ are the streamwise mean velocity and the shear direction coordinate, respectively, $l_g$ is the correlation length of the streamwise fluctuating velocities in the shear direction, and $a$ is the local sound speed. Tandra et al. [10] suggested that $l_g$ be calculated as $l_g = C_D \frac{\beta_{ij}}{\beta_{ij}}$, where $C_D$ is a closure coefficient usually considered to be 0.09.

To ensure the positivity of the turbulence kinetic energy in the vicinity of shock waves, Thivet et al. [6] suggested a realizability condition on the eddy viscosity,

$$\mu_T = \bar{\rho} \min \left( C_\mu \frac{k^2}{\varepsilon}, \sqrt{C_\mu \frac{k}{S_{Thivet}}} \right)$$  

(10)

where $S_{Thivet}$ is defined as

$$S_{Thivet} = \sqrt{2 (S_{ij} S_{ji} - \frac{1}{3} S_{ik}^2)}$$  

(11)

where $S_{ij}$ is the strain tensor.

Finally, to account for shock unsteadiness effects, unsteady motion of the shock wave front caused by the turbulence scale fluctuations, Sinha et al. [7] proposed that the turbulence production term, $P_k$, be calculated as

$$P_k = c' \mu_T (2 S_{ij} S_{ji} - \frac{2}{3} S_{ik}^2) - \frac{2}{3} \bar{\rho} k S_{kk}$$  

(12)

where

$$c' = 1 - f_s \left[ 1 + \frac{b_1 \varepsilon}{\sqrt{3 k S_{Thivet}}} \right]$$  

(13)

and

$$f_s = \frac{1}{2} \frac{1}{2} \tanh(5 S_{kk} / S_{Thivet} + 3)$$  

(14)

$S_{Thivet}$ is defined by Equation 11. Equation 14 defines $f_s$ such that it is close to 1 in highly compressed regions and close to 0 otherwise.

The original version of CFDLib included an implementation of the standard $k - \varepsilon$ model. The compressibility modifications were implemented by Tandra [10], while the other two were implemented by Emami et al. [12].

The numerical scheme implemented in CFDLib is a finite volume, explicit, cell-centered, total variable diminishing (TVD) method, that utilizes an arbitrary Lagrangian-Eulerian (ALE) time split operator for advancing the averaged flow variables. For details of the turbulence model and numerical method, refer to Ref. [12].

### 4 RESULTS

All simulations were run in a 2-D axisymmetric coordinate system. Based on the pressure ratio values, the inlet boundary conditions (velocity, density and temperature) corresponding to the nozzle exit were calculated. The ambient pressure was applied as a boundary condition far from the jet centerline. The calculations were performed on different computational domains and using different meshes, to assess both domain size and mesh dependence. Simulations were run from $t=0$ to $t=20$ msec. After about 4 msec the jet flow reached a quasi-steady form, but as expected, the time dependent oscillations in flow parameters were not damped. Data
files were dumped every $2 \times 10^{-4}$ sec, i.e. the “sampling” frequency was $1/(2 \times 10^{-4})$ Hz. Using higher sampling frequencies did not yield significantly different results.

Figure 3 (top) shows instantaneous pressure along the centerline of the jet, corresponding to 80 different times from 4 msec to 20 msec. As can be seen, for the first diameter away from the nozzle exit, the pressure does not fluctuate; this is in agreement with the measurements of Panda [1], that indicated the shock cell closest to the nozzle exit does not oscillate. Further downstream, the value of pressure at each point fluctuates with time and the thin lines, related to instantaneous pressure, form one thick line. Figure 3 (bottom) shows the time average pressure along the centerline. As shown in Figure 3, overall agreement of the simulation results with the data set of Seiner and Norum [3] is satisfactory. The discrepancies, especially in the pressure wave amplitudes, may be due to both experimental errors (especially given the unsteady nature of the flow), and the inability of RANS simulations to capture some of the physical phenomena of the flow. Similar disagreements have been reported by other researchers (e.g. [13] and [14]).

To show that the fluctuations are not limited to the centerline and occur throughout the flow field, the instantaneous and time average pressures along $r/D = 0.25$ and $r/D = 0.45$ lines are plotted in Figures 4 and 5, respectively; note that the experimental data shown in these two figures are those of Norum and Seiner [15].
To better assess these fluctuations in the time domain, it is useful to look at the root mean squared (rms) values of the flow parameters. We define a kinetic energy related to fluctuations of the flow parameters, $K_{\text{rms}}$,

$$K_{\text{rms}} = \frac{1}{2} \rho_M (u_{x,\text{rms}}^2 + u_{r,\text{rms}}^2)$$

(15)

where $\rho_M$ is the time average density, and $u_{x,\text{rms}}$ and $u_{r,\text{rms}}$ are the rms values of axial and radial flow velocities, respectively. We refer to this as the “instability wave kinetic energy”, since the fluctuations are caused by the flow instabilities.

Figure 6 shows contours of the instability wave kinetic energy, obtained from simulation. The maximum kinetic energy occurs at the jet boundary close to the nozzle exit; see the red color in the contour. This should be related to Kelvin-Helmholtz instabilities, as they occur at the boundary of the jet with the ambient fluid. In the rest of the plot, the higher kinetic energies can be seen as oblique lines. These are the oblique shock waves formed in the under-expanded jet, and indicate that the maximum fluctuations occur in the vicinity of shocks. Studying the pressure contours confirms this finding, indicating that the fluctuations result from the interaction of instability waves and shock waves: Figure 7 (top) shows the time average pressure, $P_M$, contours, and Figure 7 (bottom) shows the $rms$ pressure, $P_{\text{rms}}$, contours, normalized by the ambient pressure, $P_\infty$. The vertical dashed lines show the locations at which $P_{\text{rms}}$ is maximum; as can be seen, the maximum $P_{\text{rms}}$ correspond to the points between the compression and expansion regions, where the shock waves form. This, again, confirms that the maximum fluctuations occur in the vicinity of the shock waves.

The time dependent fluctuations can be transferred to the frequency domain using a Fourier transform. Figure 8 shows the same time average pressure contour (top), and the dominant Fourier transform of pressure (middle) and axial velocity (bottom) fluctuations along the jet centerline. It can be seen that the dominant Fourier transforms occur where the shock waves form, i.e. in front of the compression regions. This, again, confirms that the maximum fluctuations occur close to shock waves.

Finally, Figure 9 shows the Fourier transform of pressure fluctuation at one point in the physical domain ($x/D=5$, $r/D=0.5$), versus frequency. As can be seen, the dominant Fourier transforms correspond to two distinct frequency ranges. These two distinct frequency ranges may be related to two instabilities: Kelvin-Helmholtz, and acoustic instability waves. These results are qualitatively in agreement with the results of the linear instability theory. Studying the spectrum at other points in the physical domain yielded similar results.

Future work includes filtering these dominant frequencies (one at a time, as illustrated in Figure 10), and then transferring the results back to the time domain, to confirm that the two dominant frequency ranges are related to the Kelvin-Helmholtz and acoustic instability waves.

REFERENCES


Figure 8: Time average pressure contour (top), and distribution of the dominant Fourier transform of pressure (middle) and axial velocity (bottom) along the jet centerline.

Figure 9: Pressure fluctuations in the frequency domain: Fourier transform at $x/D=5$, $r/D=0.5$.

Figure 10: Filtration of the pressure spectrum.


